Solution to the Problem of Maximum Volume with Minimum Cost

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Math S-599

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Outline

1 Define Variables

2 Compute Surface Area and Volume

3 Find Optimal Dimensions

4 Check Answer

Define Variables

Let

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- w be the width of the box

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- w be the width of the box
- h be the height of the box

Define Variables

We know the box has a square base, thus $w = l$.
Let SA be the surface area of the box and V be the volume.

Relationship from Surface Area

Using the given surface area to establish relationship between w and h.

$$SA = 432 = 2(ww + wh + hw)$$

$$432 = 2w^2 + 4wh$$

$$h = \frac{432 - 2w^2}{4w} \qquad \text{isolate h}$$

$$h = \frac{108}{w} - \frac{w}{2}$$

Compute Volume

Now compute volume using the relationship between w and h to simplify the expression of volume with two unknown variables (w and h) to one unknown variable w.

$$\begin{aligned} V &= w^2 h \\ &= w^2\left(\frac{108}{w} - \frac{w}{2}\right) && \text{substitute } h \\ &= 108w - \frac{w^3}{2} \end{aligned}$$

Find Optimal Dimensions

Use First Derivative Test. Find the critical points first.

$$\frac{dV}{dw} = 0 = 108 - \frac{3w^2}{2}$$
$$108 = \frac{3w^2}{2}$$
$$w = \pm\sqrt{72}$$

We can discard negative critical points since we are dealing with dimensions.

Check if $w = \sqrt{72}$ is the width that helps the box to achieve its maximum volume. Since $V'(w = 8) = 12 > 0$ and $V'(w = 9) = -13.5 < 0$, we conclude that volume of the box is increasing when $w < \sqrt{72}$, and is decreasing when $w > \sqrt{72}$. Volume of the box achieves maximum at $w = \sqrt{72}$.

Look Back to Check Your Answer

When $w = \sqrt{72}$, $h = \sqrt{72}$. The box should be formed as a cube to achieve the maximum volume. **But does this answer make sense in the real world application? Why?**